PERTH MODERN SCHOOL
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Semester One Examination, 2018

Question/Answer booklet

MATHEMATICS UNIT

Section One: Calculator-free

Your Name Solutions

Your Teacher's Name ______________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Mark	Question	Mark
1		5	
2		6	
3		7	
4			

Section One: Calculator-free **(52 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

Working time: 50 minutes.

Question 1 **(~~8~~ 9 marks)**

(a) Hilbert numbers are defined as numbers that are 1 greater than the multiples of 4, such as 5, 9, 13, 17, 21, 25……. Prove that the product of two Hilbert numbers is a Hilbert number. (4 marks)

Let the two Hilbert numbers be $4m+1$, $4n+1$, $m, n \in \mathbb{Z}$

$(4m+1)(4n+1)$

$= 16mn + 4m + 4n + 1$

$= 4(mn + m + n) + 1$

Therefore the product is a Hilbert number.

(b) Prove that if a square and a circle have the same perimeter P, the circle will have the greater area. (~~4~~ 5 marks)

Let the side length of a square be "a".
the radius of a circle be "r."

$4a = P \quad \therefore a = \frac{P}{4}$

$2\pi r = P \quad r = \frac{P}{2\pi}$

Area of square $= \left(\frac{P}{4}\right)^2 = \frac{P^2}{16}$

Area of circle $= \pi\left(\frac{P}{2\pi}\right)^2 = \frac{P^2}{4\pi}$

$\frac{P^2}{16} < \frac{P^2}{4\pi}$ as $16 > 4\pi$

$\therefore$ Circle will have a greater area.

Question 2 **(10 marks)**

Given that $\boldsymbol{a} = -2\boldsymbol{i} + 8\boldsymbol{j}$ and $\boldsymbol{b} = 5\boldsymbol{i} - 6\boldsymbol{j}$, find:

(a) $|\boldsymbol{a} + \boldsymbol{b}|$. (3 marks)

$$|a+b| = |3i+2j| = \sqrt{3^2+2^2} = \sqrt{13}$$ ✓ ✓ ✓

(b) the unit vector parallel to $\boldsymbol{a} + \boldsymbol{b}$. (1 mark)

$$\frac{a+b}{\sqrt{a^2+b^2}} = \frac{3i+2j}{\sqrt{13}}$$ ✓

(c) a vector that is parallel to $\boldsymbol{a} + \boldsymbol{b}$ but with a magnitude of 5. (1 mark)

$$5 \times \frac{a+b}{a^2+b^2} = \frac{5}{\sqrt{13}}(3i+2j)$$ ✓

(d) $\boldsymbol{a}$ in terms of $\boldsymbol{p}$ and $\boldsymbol{q}$ where $\boldsymbol{p} = 2\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{q} = -3\boldsymbol{i} + 3\boldsymbol{j}$. (5 marks)

$a = \lambda p + \gamma q$

$-2i + 8j = \lambda(2i + j) + \gamma(-3i + 3j)$ ✓

$$\begin{cases} 2\lambda - 3\gamma = -2 & (1) \\ \lambda + 3\gamma = 8 & (2) \end{cases}$$ ✓

$(1) + (2) \quad 3\lambda = 6$

$\therefore \lambda = 2$ ✓

Sub in (2): $2 + 3\gamma = 8$

$\gamma = 2$ ✓

$\therefore a = 2p + 2q.$ ✓

Question 3 **(4 marks)**

Use *Proof by Contradiction* to show that there is no positive integers a and b, such that

$$a^2 - 4b^2 = 1$$

Assume $\exists\, a, b \in \mathbb{Z}^*$ s.t. $a^2 - 4b^2 = 1$ ✓

$(a+2b)(a-2b) = 1$

$\begin{cases} a+2b = 1 \\ a-2b = 1 \end{cases}$ OR $\begin{cases} a+2b = -1 \\ a-2b = -1 \end{cases}$

$\therefore \begin{cases} a = 1 \\ b = 0 \end{cases}$ ✓ $\therefore \begin{cases} b = 0 \\ a = -1 \end{cases}$

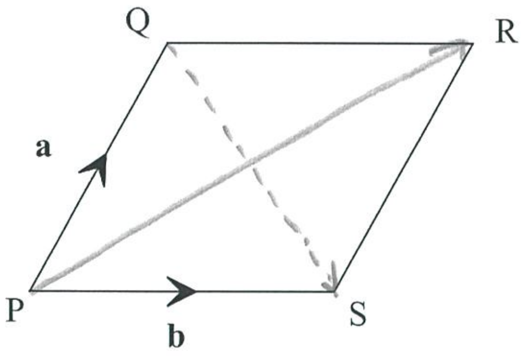
which contradicts $b \in \mathbb{Z}^*$ ✓ which contradicts $a, b \in \mathbb{Z}^*$

Therefore there is no $a, b \in \mathbb{Z}^*$ s.t. $a^2 - 4b^2 = 1$ ✓

Question 4 **(6 marks)**

$PQRS$ is a rhombus. $\boldsymbol{PQ} = \boldsymbol{a}$ and $\boldsymbol{PS} = \boldsymbol{b}$.

(a) Find $\boldsymbol{PR}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. (1 mark)



$\overrightarrow{PR} = \tilde{a} + \tilde{b}$ ✓

(b) Find $\boldsymbol{QS}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. (1 marks)

$\overrightarrow{QS} = \tilde{b} - \tilde{a}$ ✓

(c) Hence, use a vector method to show that the diagonals of a rhombus are perpendicular to each other. (4 marks)

$\overrightarrow{PR} \cdot \overrightarrow{QS} = (\tilde{a}+\tilde{b})(\tilde{b}-\tilde{a})$ ✓

$= \tilde{b}^2 - \tilde{a}^2$

$= |b|^2 - |a|^2$ ✓

$= 0$ ✓ as $PQRS$ is a rhombus

$\therefore \overrightarrow{PR} \perp \overrightarrow{QS}$ ✓

Question 5 **(7 marks)**

(a) Prove the recurring decimal $3.2\overline{10}$ is rational by expressing it as a fraction. (3 marks)

Let $x = 3.2\overline{10}$

$10x = 32.\overline{10}$ ①

$1000x = 3210.\overline{10}$ ② ✓

② − ① $990x = 3178$ ✓

$$x = \frac{3178}{990} \text{ or } 3\frac{208}{990}$$

$$\left(\frac{1589}{495}\right) \text{ or } \left(3\frac{104}{495}\right)$$ ✓

(b) Prove irrationality by contradiction for $\sqrt{10}$. (4 marks)

Assume $\sqrt{10}$ is rational which can be expressed as a simplified fraction $\frac{a}{b}$. (a, b have no common factor) ✓

$\frac{a}{b} = \sqrt{10}$

$\frac{a^2}{b^2} = 10$

$a^2 = 10b^2$

$\therefore a^2$ is a multiple of 10

$\therefore a$ is a multiple of 10. ✓

$\therefore a = 10k$

$a^2 = 100k^2$

$\frac{a^2}{b^2} = \frac{100k^2}{b^2} = 10$

$\therefore b^2 = 10k^2$

$\therefore b^2$ is a multiple of 10

b is a multiple of 10 ✓

which means a, b have a common factor of 10, contradicting the assumption ✓

Therefore, $\sqrt{10}$ is irrational

1' assumption

2' show a is multiple of 10

3' show b is multiple of 10

4' State contradiction

Question 6 **(8 marks)**

(a) Show that the second number in row n of Pascal's triangle is always n for $n > 0$. (2 mark)

$$^nC_1 = \frac{n!}{(n-1)!\,1!} = \frac{n \times (n-1)!}{(n-1)!} = n$$

(b) Prove that the third number of any row of Pascal's triangle is always $\frac{1}{2}\,n\,(n-1)$. (2 mark)

$$^nC_2 = \frac{n!}{(n-2)!\,2!} = \frac{n \times (n-1) \times (n-2)!}{(n-2)! \times 2}$$

$$= \frac{n \times (n-1)}{2}$$

(c) Show that for any row in Pascals' Triangle, the $(r+1)$th number is $\frac{n-r}{r+1}$ times the rth number, that is, $^nC_{r+1} = \frac{n-r}{r+1} \times {}^nC_r$ assuming that $^nC_r = 1$ and $n > 0$. (4 marks)

$$LHS = {}^nC_{r+1} = \frac{n!}{(n-(r+1))!\,(r+1)!}$$

$$= \frac{n!}{(n-r-1)!\,(r+1) \times (r)!}$$

$$= \frac{(n-r) \times n!}{(n-r) \times (n-r-1)!\,(r+1) \times r!}$$

$$= \frac{(n-r) \times n!}{(n-r)! \times (r+1) \times r!}$$

$$= \frac{n-r}{r+1} \times \frac{n!}{(n-r)!\,r!} = \frac{n-r}{r+1} \times {}^nC_r = RHS$$

Question 7 **(8 marks)**

(a) Prove that any 3-digit number is divisible by three if the sum of its digits is divisible by three. [Hint: let the units digit be *a*, the tens digit be *b*, and the hundreds digit be *c*.]

(3 marks)

Let the 3-digit number be

$100c + 10b + a$ ✓ where $a+b+c = 3m$.

$= 99c + 9b + c + b + a$

$= 3(33c+3b) + 3m$ ✓

$= 3(33c+3b+m)$ ✓

hence divisible by 3.

End of questions

(b) Use mathematical induction to prove that $2^{6n} + 3^{2n-2}$ is always divisible by 5, for $n \in N$.

(5 marks)

$n = 1$, $2^6 + 3^0 = 65$ is divisible by 5.

Assume true for $n = k$, i.e. $2^{6k} + 3^{2k-2} = 5m$.

$2^{6k} = 5m - 3^{2k-2}$

For $n = k+1$, the expression gives

$2^{6(k+1)} + 3^{2(k+1)-2} = 2^6 \times 2^{6k} + 3^{2k}$

$= 64 \times (5m - 3^{2k-2}) + 3^{2k-2} \times 3^2$

$= 64 \times 5m - 64 \times 3^{2k-2} + 3^{2k-2} \times 9$

$= 5(64m) - 3^{2k-2} \times (64 - 9)$

$= 5 \times 64m - 3^{2k-2} \times 55$

$= 5(64m - 3^{2k-2} \times 11)$

Hence divisible by 5.

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Semester One Examination, 2018

Question/Answer booklet

MATHEMATICS SPECIALIST UNIT 1

Section Two: Calculator-assumed

Your Name Solutions

Your Teacher's Name ______________________

Time allowed for this section

Reading time before commencing work: ten minutes
Working time: one hundred minutes

Materials required/recommended for this section

To be provided by the supervisor

This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate

Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators approved for use in this examination

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Question	Marks	Question	Marks
8		15	
9		16	
10		17	
11		18	
12		19	
13		20	
14			

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	7	7	50	52	35
Section Two: Calculator-assumed	13	13	100	94	65
				Total	100

Instructions to candidates

1. The rules for the conduct of the Western Australian Certificate of Education ATAR course examinations are detailed in the *Year 12 Information Handbook 2016*. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet.

3. You must be careful to confine your answers to the specific questions asked and to follow any instructions that are specific to a particular question.

4. Additional pages for the use of planning your answer to a question or continuing your answer to a question have been provided at the end of this Question/Answer booklet. If you use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number.

5. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

6. It is recommended that you **do not use pencil**, except in diagrams.

7. The Formula sheet is **not** to be handed in with your Question/Answer booklet.

Section Two: Calculator-assumed **(94 Marks)**

This section has **Thirteen (13)** questions. Answer **all** questions. Write your answers in the spaces provided.

Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer.

- Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
- Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question that you are continuing to answer at the top of the page.

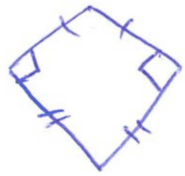
Working time: 100 minutes

Question 8 **(6 marks)**

Determine whether each of the following statement is true or false. Prove in general if the statement is correct. Use counter examples if the statement is false.

(a) A quadrilateral with a right angle must be a rectangle.

False. ✓ A kite with two ✓ right-angles.



(b) For all positive numbers a, $a^2 \leq a^3$

False. ✓ $a = \frac{1}{2}$ $a^2 > a^3$.

$a^2 = \frac{1}{4}$ ✓

$a^3 = \frac{1}{8}$

(c) $x^2 + x + 11$ always gives a prime number.

False. ✓ $11^2 + 11 + 11 = 11(11+1+1)$

$x = 11,$ $= 11 \times 13$

not prime. ✓

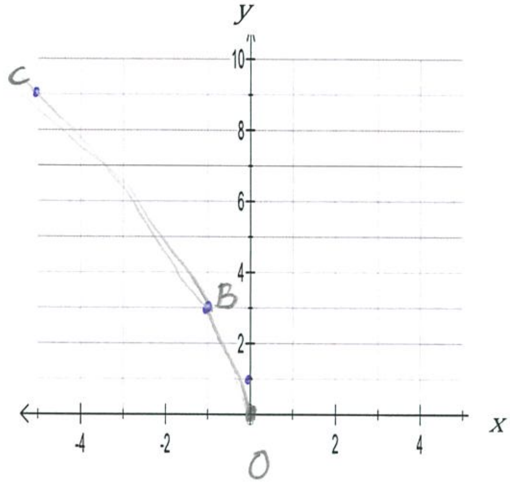
Question 9 **(8 marks)**

(a) The position vectors of points A, B, and C are $\boldsymbol{i}$, $-\boldsymbol{i} + 3\boldsymbol{j}$ and $-5\boldsymbol{i} + 9\boldsymbol{j}$ respectively. Show that the points A, B, and C are collinear and hence state the ratio of $AB : BC$. (4 marks)

$\overrightarrow{AB} = \begin{pmatrix} -2 \\ 3 \end{pmatrix}$ ✓ $\overrightarrow{BC} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$ ✓

$\overrightarrow{BC} = 2\overrightarrow{AB}$ ✓ hence collinear.

$AB : BC = 1 : 2$ ✓



(b) Using the information in part (a). It is given that $OBCD$ is a parallelogram and that E is the point such that $\overrightarrow{DB} = \frac{1}{3}\overrightarrow{ED}$. Find the position vectors of D and E. (4 marks)

OBCD is a parallelogram.

$\overrightarrow{OD} = BC = -4i + 6j$ ✓

$\overrightarrow{BD} = \overrightarrow{BO} + \overrightarrow{OD}$

$= \begin{pmatrix} 1 \\ -3 \end{pmatrix} + \begin{pmatrix} -4 \\ 6 \end{pmatrix}$

$= \begin{pmatrix} -3 \\ 3 \end{pmatrix}$ ✓

$\overrightarrow{OE} = \overrightarrow{OD} + \overrightarrow{DE}$

$= \overrightarrow{OD} + 3\overrightarrow{BD}$ ✓

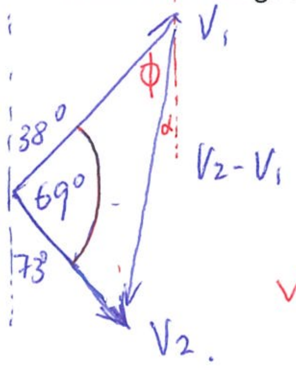
$= \begin{pmatrix} -4 \\ 6 \end{pmatrix} + 3\begin{pmatrix} -3 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} -13 \\ 15 \end{pmatrix}$

$\therefore \overrightarrow{OE} = -13i + 15j$ ✓

Question 10 **(12 marks)**

(a) The velocity of a boat v_1 changes from 25 knots at a bearing of 38° to v_2 which is 15 knots at a bearing of 107°. What is the change in the velocity? Draw a diagram for this situation. State the changes in the speed and the direction. [Hint: find $v_2 - v_1$.] (5 marks)



$$|V_2 - V_1| = \sqrt{25^2 + 15^2 - 2 \times 25 \times 15 \times \cos 69°}$$

$$= 24.108. \checkmark$$

$\theta = 107° - 38° = 69°. \checkmark$

$$\phi = \cos^{-1}\left(\frac{25^2 + 24.108^2 - 15^2}{2 \times 25 \times 24.108}\right)$$

$$= 35.5° \checkmark$$

$\alpha = 38° - 35.5 = 2.5°$

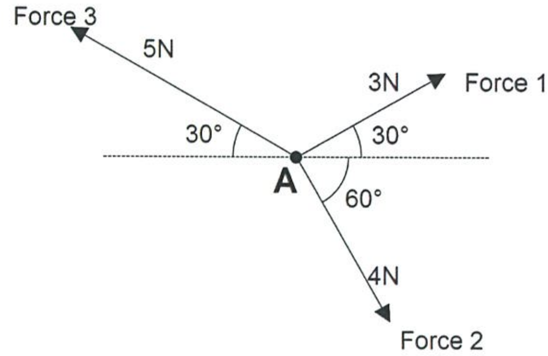
bearing $= 180° + 2.5° = 182.5°. \checkmark$

✓ diagram

✓✓ magnitude / speed

✓✓ direction / bearing

(b) Three forces act on the point *A* as shown. What is the magnitude and direction of the resultant force acting on *A*? (7 marks)



$F_1 = 3\cos 30^\circ i + 3\sin 30^\circ j$

$F_2 = 4\cos 60^\circ i - 4\sin 60^\circ j$

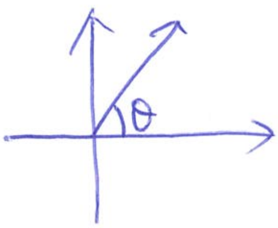
$F_3 = -5\cos 30 i + 5\sin 30^\circ j$

$F_r = F_1 + F_2 + F_3$

$= (2-\sqrt{3})i + (4-2\sqrt{3})j$

$\approx 0.2679 i + 0.5359 j$

$|F_r| = 0.6N$



$\tan\theta = \dfrac{0.5359}{0.2679}$

$\theta = 63.43^\circ$

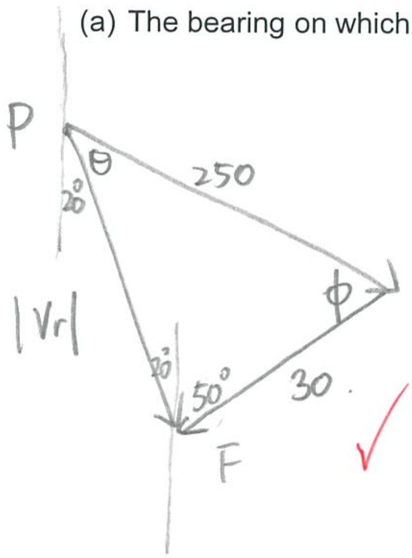
$\therefore$ bearing is 26.57°

magnitude is 0.6N.

Question 11 **(8 marks)**

An aircraft is to be flown from Perth (P) to a bush fire (F) that is 180 km away on bearing of 160°. If a wind of 30km/hr is blowing **from** a bearing of 50° and the aircraft can maintain a steady speed of 250km/hr in still air find:

(a) The bearing on which the plane must be set (Draw a diagram) (4 marks)



$$\frac{\sin\theta}{30} = \frac{\sin 70^\circ}{250}$$

$$\theta = 6.47^\circ$$

$$\therefore \text{Bearing} = 160^\circ - 6.47^\circ$$
$$= 153.53^\circ$$

(b) The actual speed of the plane (2 marks)

$$\phi = 180 - 70 - 6.47 = 103.53^\circ$$

$$\frac{|V_r|}{\sin 103.53^\circ} = \frac{250}{\sin 70^\circ} \quad \therefore |V_r| = 258.7 \text{ km/h.}$$

(c) The time of the journey (2 marks)

$$\text{time} = \frac{180}{258.7} = 0.7\text{hr} \quad \text{or } 42 \text{ min}$$

Question 12 **(10 marks)**

(a) Find the values of λ and μ given that $\boldsymbol{a}$ and $\boldsymbol{b}$ are non-parallel vectors and (4 marks)

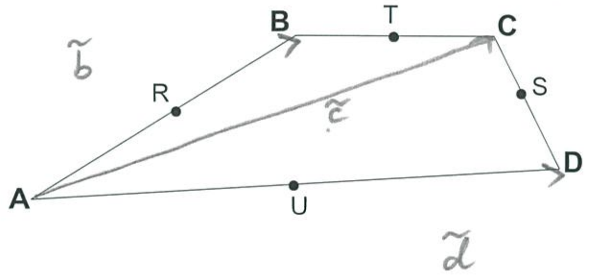
$$2\lambda\boldsymbol{a} + 3\mu\boldsymbol{a} - \mu\boldsymbol{b} + 2\boldsymbol{b} = \lambda\boldsymbol{b} + 2\boldsymbol{a}$$

$$(2\lambda + 3\mu)a + (2 - \mu b) = 2a + \lambda b \checkmark$$

$$2\lambda + 3\mu = 2$$
$$2 - \mu = \lambda \checkmark$$

$$\lambda = 4 \checkmark$$
$$\mu = -2 \checkmark$$

(b) In the quadrilateral $ABCD$ shown below $\overrightarrow{AB} = b,\ \overrightarrow{AC} = c,\ \overrightarrow{AD} = d$. The points R, T, S and U are the midpoints of the sides shown in the diagram. Show, using vectors $\boldsymbol{b}, \boldsymbol{c}$ *and* $\boldsymbol{d}$ that $\overrightarrow{TS}$ is parallel to $\overrightarrow{RU}$. (6 marks)



$$\overrightarrow{BC} = \tilde{c} - \tilde{b}$$
$$\overrightarrow{TC} = \tfrac{1}{2}(\tilde{c} - \tilde{b}) \checkmark$$
$$\overrightarrow{CD} = \tilde{d} - \tilde{c}$$
$$\overrightarrow{CS} = \tfrac{1}{2}(\tilde{d} - \tilde{c}) \checkmark$$
$$\overrightarrow{TS} = \overrightarrow{TC} + \overrightarrow{CS}$$
$$= \tfrac{1}{2}c - \tfrac{1}{2}b + \tfrac{1}{2}d - \tfrac{1}{2}c$$
$$= \tfrac{1}{2}(\tilde{d} - \tilde{b}) \checkmark$$

$$\left.\begin{aligned}\overrightarrow{AR} &= \tfrac{1}{2}\tilde{b} \\ \overrightarrow{AU} &= \tfrac{1}{2}d\end{aligned}\right\} \checkmark$$

$$\overrightarrow{RU} = \overrightarrow{AU} - \overrightarrow{AR} = \tfrac{1}{2}(\tilde{d} - \tilde{b}) \checkmark$$

$$\therefore \overrightarrow{TS} = \overrightarrow{RU} \checkmark$$

hence $\overrightarrow{TS} \parallel \overrightarrow{RU}$

Question 13 **(8 marks)**

(a) For digits {1, 2, 3, 4, 5, 6}

(i) How many numbers can be formed in total, if each digit is only used at most once?
[Hint: consider all 1 to 6-digit numbers]. (2 marks)

$$^6P_1 + {}^6P_2 \times 2! + {}^6P_3 \times 3! + {}^6P_4 \times 4! + {}^6P_5 \times 5! + {}^6P_6 \times 6!$$ ✓

$= 1956$ ✓

(ii) How many of the above are less than 600,000? (2 marks)

Number above 600,000 $= 5!$ ✓

$\therefore$ Number below 600,000 $= 1956 - 5!$
$= 1836$ ✓

(b) For digits {1, 2, 3, 4, 5, 5, 6, 6}, how many distinct four-digit numbers can be formed? (4 marks)

- All diff digits : $\frac{6!}{2!} = 360$ ✓
- Including two "5"s but no more than one "6": $\binom{4}{2} \times 5 \times 4 = 120$
- Including two "6"s but no more than one "5": $\binom{4}{2} \times 5 \times 4 = 120$. ✓
- Including two "5"s and two "6"s : $\binom{4}{2} = 6$ ✓

Total : $360 + 120 + 120 + 6 = 606$ ✓

Question 14 **(5 marks)**

A set S contains all the integers between 3 and 122 inclusive.

(a) how many numbers in set S are multiples of 6? (1 mark)

$120 \div 6 = 20$ ✓

(b) how many numbers in set S are multiples of 6 or 8? (2 marks)

$120 \div 8 = 15$ $\quad$ $LCM(6,8) = 24$

$120 \div 24 = 5$ ✓

$20 + 15 - 5 = 30$ ✓

(c) how many numbers in set S are multiples of either 6 or 8 but not both? (2 marks)

$20 + 15 - 5 - 5 = 25$ ✓ ✓

Question 15 **(3 marks)**

Show that if 50 different integers are selected from the set {1, 2, 3, ..., 97, 98}, there will be at least two integers whose sum is 99.

Create pigeonholes using the sets {1, 98}, {2, 97} ... {49, 50}. In total, 49 sets. ✓

Since there're 50 numbers (pigeons) ✓ by Pigeonhole Principle there must be at least two numbers in the same pigeonhole, and each pair of numbers add up to 99. ✓

Question 16 **(6 marks)**

(a) Determine the angle between the vectors $< -12, 7 >$ and $< 3, 8 >$. (2 marks)

$$\cos\theta = \frac{\begin{pmatrix}-12\\7\end{pmatrix}\cdot\begin{pmatrix}3\\8\end{pmatrix}}{\sqrt{12^2+7^2}\cdot\sqrt{3^2+8^2}} = 0.1685$$

$$\theta = 80.3^\circ \text{ (classpad)}$$

(b) Determine the value of a so that the vectors $< 7, a >$ and $< 10, 4 >$ are perpendicular. (2 marks)

$$\begin{pmatrix}7\\a\end{pmatrix}\cdot\begin{pmatrix}10\\4\end{pmatrix} = 0$$

$$70 + 4a = 0$$

$$a = -\frac{70}{4}$$

(c) Determine the exact scalar projection of $< 3, -5 >$ on $< -8, 4 >$. (2 marks)

$$\left|\begin{pmatrix}3\\-5\end{pmatrix}\right|\cdot\cos\theta = \frac{\begin{pmatrix}3\\-5\end{pmatrix}\begin{pmatrix}-8\\4\end{pmatrix}}{\left|\begin{pmatrix}-8\\4\end{pmatrix}\right|} = \frac{-44}{\sqrt{80}} = -\frac{11}{\sqrt{5}}$$

Question 17 **(10 marks)**

How many ways can the letters of the word ***TRIANGLE*** be arranged

(a) if there is no restriction? (1 mark)

$8! = 40320$ ✓

(b) if the first three letters must be ***RAN*** (in that <u>exact</u> order)? (2 marks)

$\underline{1}\ \underline{1}\ \underline{1}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}$

$5! = 120$ ✓ ✓

(c) if the first three letters must be ***RAN*** (in <u>any</u> order) and the next three letters must be ***TRI*** (in <u>any</u> order)? (2 marks)

$(\underline{3}\ \underline{2}\ \underline{1})(\underline{3}\ \underline{2}\ \underline{1})\underline{2}\ \underline{1}$

$3! \times 3! \times 2!$ ✓

$= 72$ ✓

(d) if the order and the placement of the vowels ***IAE*** cannot be changed? (i.e. ***IAE*** stay where they are as in *TR<u>IA</u>NGL<u>E</u>*) (2 marks)

$\underline{5}\ \underline{4}\ \underline{1}\ \underline{1}\ \underline{3}\ \underline{2}\ \underline{1}\ \underline{1}$

$5! = 120$ ✓ ✓

(e) if the order of the vowels ***IAE*** cannot be changed, though their placement may (***<u>IAE</u>***TRNGL and TR***<u>IA</u>***NG***<u>E</u>***L are acceptable but ***<u>EIA</u>***TRNGL and TR***<u>IE</u>***NGL***<u>A</u>*** are not)? (3 marks)

$\binom{8}{3} \times 5! = 6720$ ✓ ✓

Question 18 **(6 marks)**

Airports A and B are such that $\overrightarrow{AB} = 250i - 750j$ km. An aircraft is to be flown directly from A to B. In still air the aircraft can maintain a steady speed of 400km/hr. There is a wind blowing with velocity $13i + 9j$ km/hr.

(1) Find, in the form $ai + bj$, the velocity vector the pilot should set so that this velocity, together with the wind, causes the plane to travel directly from A to B. (4 marks)

$$(ai + bj) + (13i + 9j) = \lambda(250i - 750j)$$

$$\frac{a+13}{b+9} = \frac{250}{-750} = \frac{1}{-3}$$

$$b + 9 = -3a - 39$$

$$b = -3a - 48$$

$$a^2 + b^2 = 400^2$$

$$\begin{cases} a = 112 \\ b = -384 \end{cases} \text{ or } \begin{cases} a = -140.8 \\ b = 374.4 \end{cases}$$

$$V = 112i - 384j$$

(2) If the wind remains unchanged, in the form $ai + bj$, the velocity vector the pilot should now set to return directly from B to A. (2 marks)

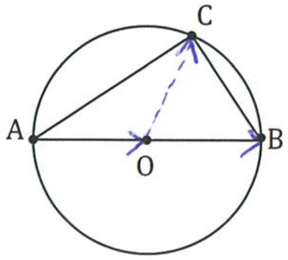
$$a = -140.8$$

$$b = 374.4$$

$$V = -140.8i + 374.4j$$

Question 19 **(4 marks)**

Use a vector method to prove that the angle in a semi-circle is a right-angle.



Hint: Let $\overrightarrow{OC} = \mathbf{c}$ and $\overrightarrow{OB} = \mathbf{b}$.

$\overrightarrow{OC} = \underset{\sim}{c}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $\overrightarrow{AO} = \overrightarrow{OB} = \underset{\sim}{b}$

$\overrightarrow{BC} = \underset{\sim}{c} - \underset{\sim}{b}$ ✓

$\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = \underset{\sim}{b} + \underset{\sim}{c}$ ✓

$\overrightarrow{AC} \cdot \overrightarrow{BC} = (\underset{\sim}{c} - \underset{\sim}{b})(\underset{\sim}{c} + \underset{\sim}{b})$ ✓

$= c^2 - b^2$

$= |c|^2 - |b|^2$

$= 0$ ✓ as $|c| = |b|$ = radius of circle.

Question 20 **(8 marks)**

(a) Show that $\frac{1}{(k+2)!} - \frac{k+1}{(k+3)!} = \frac{2}{(k+3)!}$ (2 marks)

$$LHS = \frac{k+3}{(k+3)(k+2)!} - \frac{k+1}{(k+3)!} = \frac{2}{(k+3)!} = RHS$$

(b) Prove by induction that, for all positive integers n

$$\frac{1\times2^1}{(1+2)!} + \frac{2\times2^2}{(2+2)!} + \frac{3\times2^3}{(3+2)!} + \cdots + \frac{n\times2^n}{(n+2)!} = 1 - \frac{2^{n+1}}{(n+2)!}$$ (6 marks)

$n = 1$, $LHS = \frac{1\times2}{3!} = \frac{2}{6} = \frac{1}{3}$

$RHS = 1 - \frac{2^2}{3!} = 1 - \frac{4}{6} = \frac{1}{3}$, $\therefore$ true for $n = 1$

Assume true for $n = k$, $\frac{1\times2^1}{(1+2)!} + \cdots + \frac{k\times2^k}{(k+2)!} = 1 - \frac{2^{k+1}}{(k+2)!}$

Need, $1 - \frac{2^{k+2}}{(k+3)!}$

$n = k+1$, $1 - \frac{2^{k+1}}{(k+2)!} + \frac{(k+1)\times2^{k+1}}{(k+3)!}$

$= 1 - 2^{k+1}\left(\frac{1}{(k+2)!} - \frac{k+1}{(k+3)!}\right)$

$= 1 - 2^{k+1}\left(\frac{2}{(k+3)!}\right)$ from (a)

$= 1 - \frac{2^{k+2}}{(k+3)}$

Hence, true for $n = k+1$.

Therefore by proof by induction, if true for $n=1$, true for $n = k+1$, it must be true for $n = k$, proved.

End of Questions

See next page